An Improved Convergent Iterative Method for Finding the Moore-Penrose Inverse

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Abstract : The goal of this paper is to suggest and establish an efficient iterative method based on matrix by matrix multiplications for finding the approximate inverse of non-singular square matrices. We then analytically extend this proposed method so as to compute the Moore-Penrose generalized inverse of a non-square matrix. A theoretical analysis has been employed to compare the computational efficiency of the presented scheme with the other existing methods in the literature to show that it is economic. Numerical experiments are also executed to manifest its superiority.

Keywords: Moore-Penrose inverse; Computational efficiency index; Informational efficiency index; Sparse matrices; Initial approximation.

I. INTRODUCTION

The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$ (also called pseudoinverse), denoted by $A^{\dagger}$, is a matrix $V \in \mathbb{C}^{n\times m}$, satisfying the following system of equations.

$$AVA = A, \quad VAX = X, \quad (AV)^* = AV, \quad (VA)^* = VA,$$

where A^* stands for the conjugate transpose of a matrix A.

It is well known that for any matrix $A \in \mathbb{C}^{m\times n}$, its Moore- Penrose inverse exists uniquely. In fact, it is a generalization of the inverse of a non-singular matrix which plays an important role in various fields, such as eigenvalue problems, solution to various systems of linear equations, linear least square problems, and so on, see [4]. Of course, many important approaches for computing the Moore-Penrose generalized inversion have been developed. In these methods, direct methods usually tend to require a predictable amount of resources in terms of time and storage, which normally put them out of interest especially for the cases when A is sparse. At this time, iterative methods of the class of Schulz-type can be taken into account.

The well-known second order technique of this type is the Schulz method [11] defined by

$$V_{k+1} = V_k(2I - AV_k), \quad k = 0, 1, 2, \ldots . \quad (1.1)$$

for finding the Moore-Penrose inverse. This scheme has interesting features of being based exclusively on matrix-matrix operations. The Schulz iteration has polylogarithmic complexity and is numerically stable. Let us review some of the higher-order iterative methods for Moore-Penrose inversion. The perception and reason of constructing these higher-order schemes is that (1.1) is too slow at the beginning of the process before arriving at the convergence phase for general matrices, and this would increase the computational load of the whole algorithm used for Moore-Penrose inversion.

Li et al. in [5] gave the following third order iterative convergent scheme, which is known as Chebyshev's method

$$V_{k+1} = V_k(3I - AV_k(3I - AV_k)), \quad (1.2)$$

for k = 0, 1, 2,....... In fact, a general procedure for constructing such schemes for matrix inversion had been brought forward in [7]. For instance, a high-order method of convergence order ten can be deduced as

$$V_{k+1} = V_k(I + R_k(I + R_k(I + R_k(I + R_k(I + R_k(I + R_k(I + R_k(I + R_k(I + R_k))))))))), \quad (1.3)$$

where $R_k = I - AV_k$ and k= 0,1,2..... It must be noted that in such constructions, each p-th order method, as the above one, needs exactly p times of matrix by matrix multiplications for finding the Moore-Penrose inverse.

In this paper, we present a new iterative method of the Schulz-type with tenth order of convergence for computing the Moore-Penrose inverse, but with less number of matrix by matrix multiplications (mmms) than fifteen. This would make the method quite computationally efficient in contrast to the existing iterative methods of the same type, since the governing cost in implementing the Schulz-type methods is the cost of mmms. The remaining sections of this paper unfold the contents in what follows. Section 2 is devoted to the analysis of convergence which shows that the method can be considered for the pseudoinverse as well. Section 3 infer that the new method is computationally economic. Subsequently, the method is examined in Section 4 numerically. Finally to end this paper, conclusion will be drawn in section 5.

II. A NOVEL METHOD

In this section, we present a new higher order iterative method whereas the number of mmms is lower than that corresponding method from the general schemes of [3,7]. Towards this aim, we suggest our proposed method as follows:

$$V_{k+1} = \frac{1}{32}V_k(400I - 2320(AV_k) + 8280(AV_k)^2 - 20330(AV_k)^3 + 36365(AV_k)^4 - 48940(AV_k)^5 + 50445(AV_k)^6 - 40140(AV_k)^7 + 24650(AV_k)^8 - 11584(AV_k)^9 + 4090(AV_k)^{10} - 1050(AV_k)^{11} + 185(AV_k)^{12} - 20(AV_k)^{13} + (AV_k)^{14}). \quad (2.1)$$

Using proper factorization, we attain the following efficient matrix iterative method:

$$\zeta_k = 5I + \psi_k(-4I + \psi_k),$$
$$\kappa_k = \psi_k\zeta_k,$$
$$V_{k+1} = \frac{1}{32}V_k\zeta_k(80I + \kappa_k(-80I + \kappa_k(40I + \kappa_k(-10I + \kappa_k)))), \quad (2.2)$$

wherein $\psi_k = AV_k$ and k = 0, 1, 2, : : : . Note that the method (2.2) needs seven mmms to achieve high convergence rate ten. This fact is about to be theoretically obtained in the following subsection.

Theorem 2.1. Assume that $A = [a_{i,j}]_{m\times m}$ is an invertible matrix with real or complex entries. If the initial guess V_0 satisfies

$$\|I - AV_0\| < 1, \quad (2.3)$$

then, the iteration (2:2) converges to A^{-1} with at least tenth convergence order.

Proof. For the sake of simplicity, assume that $E_0 = I - AV_0$ and $E_k = I - AV_k$ stand for the symmetric residual matrix. It is straightforward to have

$$E_{k+1} = I - AV_{k+1} = I - A(\frac{1}{32}V_k(400I - 2320(AV_k) + 8280(AV_k)^2 - 20330(AV_k)^3 + 36365(AV_k)^4 - 48940(AV_k)^5 + 50445(AV_k)^6 - 40140(AV_k)^7 + 24650(AV_k)^8 - 11584(AV_k)^9 + 4090(AV_k)^{10} - 1050(AV_k)^{11} + 185(AV_k)^{12} - 20(AV_k)^{13} + (AV_k)^{14})),$$

which further implies that

$$\begin{aligned} E_{k+1} &= \frac{1}{32}(I - AV_k)^{10}(2I - AV_k)^5 \\ &= \frac{1}{32}(I - AV_k)^{10}(I + I - AV_k)^5 \\ &= \frac{1}{32}E_k^{10}(I + E_k)^5 \\ &= \frac{1}{32}E_k^{10}(1 + 5E_k + 10E_k^2 + 10E_k^3 + 5E_k^4 + E_k^5) \\ &= \frac{1}{32}(E_k^{10} + 5E_k^{11} + 10E_k^{12} + 10E_k^{13} + 5E_k^{14} + E_k^{15}). \end{aligned}$$
(2.4)

Hence by taking an arbitrary norm from both sides of (2.4), we obtain

$$\|E_{k+1}\| \le \frac{1}{32}(\|E_k\|^{10} + 5\|E_k\|^{11} + 10\|E_k\|^{12} + 10\|E_k\|^{13} + 5\|E_k\|^{14} + \|E_k\|^{15}). \quad (2.5)$$

In addition, since $\|E_0\| < 1,$ by relation (2.4) and using mathematical induction, we have the following relation

$$\|E_1\| \le \frac{1}{32}(\|E_0\|^{10} + 5\|E_0\|^{11} + 10\|E_0\|^{12} + 10\|E_0\|^{13} + 5\|E_0\|^{14} + \|E_0\|^{15}) \le \|E_0\|^{10} < 1. \quad (2.6)$$

If we take into consideration $\|E_k\| < 1,$ then

$$\|E_{k+1}\| \le \frac{1}{32}(\|E_k\|^{10} + 5\|E_k\|^{11} + 10\|E_k\|^{12} + 10\|E_k\|^{13} + 5\|E_k\|^{14} + \|E_k\|^{15}) \le \|E_k\|^{10}. \quad (2.7)$$

Furthermore, we get that

$$\|E_{k+1}\| \leq \|E_k\|^{10} \leq \ldots \leq \|E_0\|^{10^{k+1}} < 1. \quad (2.8)$$

That is, $I - AV_k \to 0$, as $k \to \infty$ and thus $V_k \to A^{-1}$, when $k \to \infty$.

Now, we must show that the tenth order of convergence is obtained for the sequence $\{V_k\}_{k=0}^{k=\infty}$. To do this, we denote $\varepsilon_k = A^{-1} - V_k$ as the error matrix in the iterative procedure (2.2). Using (2.4) we have

$$I - AV_{k+1} = \frac{1}{32}((I - AV_k)^{10} + 5(I - AV_k)^{11} + 10(I - AV_k)^{12} + 10(I - AV_k)^{13} + 5(I - AV_k)^{14} + (I - AV_k)^{15}). \quad (2.9)$$

which further implies that

$$A(A^{-1} - V_{k+1}) = \frac{1}{32}(A^{10}(A^{-1} - V_k)^{10} + 5A^{11}(A^{-1} - V_k)^{11} + 10A^{12}(A^{-1} - V_k)^{12} + 10A^{13}(A^{-1} - V_k)^{13} + 5A^{14}(A^{-1} - V_k)^{14} + A^{15}(A^{-1} - V_k)^{15}). \quad (2.10)$$

This is simplified as

$$\|\varepsilon_{k+1}\| \leq \frac{1}{32}(\|A\|^9\|\varepsilon_k\|^{10} + 5\|A\|^{10}\|\varepsilon_k\|^{11} + 10\|A\|^{11}\|\varepsilon_k\|^{12} + 10\|A\|^{12}\|\varepsilon_k\|^{13} + 5\|A\|^{13}\|\varepsilon_k\|^{14} + \|A\|^{14}\|\varepsilon_k\|^{15}). \quad (2.11)$$

And hence

$$\|\varepsilon_{k+1}\| \leq \frac{1}{32}(\|A\|^9 + 5\|A\|^{10}\|\varepsilon_k\|^1 + 10\|A\|^{11}\|\varepsilon_k\|^2 + 10\|A\|^{12}\|\varepsilon_k\|^3 + 5\|A\|^{13}\|\varepsilon_k\|^4 + \|A\|^{14}\|\varepsilon_k\|^5)\|\varepsilon_k\|^{10}. \quad (2.12)$$

The error inequality (2.12) clearly reveals that the iteration (2.2) converges with tenth order to A^{-1}. This completes the proof.

At this time, we discuss an application of (2.2) for finding the Moore-Penrose inverses. In order to validate the applicability of our proposed scheme, we must start it with a viable initial matrix. Ben-Israel and his colleagues in [1,2] used the method (1.1) with the starting value

$$V_0 = \alpha A^*, \quad (2.13)$$

where $0 < \alpha < \frac{2}{\rho(AA^*)}$ and $\rho(.)$ denotes the spectral radius.

Based on the following Lemma, we show analytically that in case of having singular or rectangular matrices, scheme (2.2) converges to the Moore-Penrose generalized inverse.

Lemma 2.2. For the sequence $\{V_k\}_{k=0}^{k=\infty}$ generated by the Schulz-type iterative method (2:2), it holds that

$$(AV_k)^* = AV_k, \quad (V_kA)^* = V_kA, \quad V_kAA^\dagger = V_k, \quad A^\dagger AV_k = V_k.$$

Proof. The proof of this lemma is based on mathematical induction. Such a process is similar to the Lemma 2.1 of [9], and it is hence omitted.

Before stating the main theorem for computing Moore-Penrose inverse, it is required to recall that for $A \in \mathbb{C}^{m\times n}$ with the singular values $\sigma_1 > \sigma_2 > \ldots \sigma_r > 0$ and the initial approximation $V_0 = \alpha A^*$ with $0 < \alpha < \frac{2}{\sigma_1^2}$, it holds that

$$\|A(V_0 - A^\dagger\| < 1. \quad (2.14)$$

We are about to use this fact in the following theorem so as to find the theoretical order of the reported method (2.2) for finding the Moore-Penrose inverse (see [18] for more details).

Theorem 2.3. For the rectangular complex matrix $A \in \mathbb{C}^{m\times n}$, with the singular values $\sigma_1 > \sigma_2 > \ldots \sigma_r > 0$ and the sequence $\{V_k\}_{k=0}^{k=\infty}$ generated by (2:2), using the initial approximation $V_0 = \alpha A^*$, the sequence converges to the Moore-Penrose inverse $A^\dagger$ with tenth order of convergence, provided that $0 < \alpha < \frac{2}{\sigma_1^2}$.

Proof. Following the Lemma 2.2, and $\mathbb{E}_k = V_k - A^\dagger$, the error matrix for finding the Moore-Penrose inverse, we have (note that $E_k = I - AV_k$)

$$A\mathbb{E}_{k+1} = AV_{k+1} - AA^{\dagger}$$
$$= AV_{k+1} - I + I - AA^{\dagger}$$
$$= -E_{k+1} + I - AA^{\dagger}$$
$$= -\frac{1}{32}(E_k^{10} + 5E_k^{11} + 10E_k^{12} + 10E_k^{13}$$
$$+ 5E_k^{14} + E_k^{15}) + I - AA^{\dagger}. \qquad (2.15)$$

On the other hand, from the properties of Moore-Penrose inverse $A^{\dagger}$, we have

$$(I - AA^{\dagger})^t = I - AA^{\dagger}, \ t = 1, 2, 3 \ldots;$$
$$(I - AA^{\dagger})A\mathbb{E}_k = 0. \qquad (2.16)$$

The use of these relationships implies that

$$A\mathbb{E}_{k+1} = \frac{1}{32}(-(A\mathbb{E}_k)^{10} + 5(A\mathbb{E}_k)^{11}$$
$$- 10(A\mathbb{E}_k)^{12} + 10(A\mathbb{E}_k)^{13}$$
$$- 5(A\mathbb{E}_k)^{14} + (A\mathbb{E}_k)^{15}). \qquad (2.17)$$

So, for any matrix norm $\|\cdot\|$, we obtain

$$\|A\mathbb{E}_{k+1}\| \leq \frac{1}{32}(\|A\mathbb{E}_k\|^{10} + 5\|A\mathbb{E}_k\|^{11}$$
$$+ 10\|A\mathbb{E}_k\|^{12} + 10\|A\mathbb{E}_k\|^{13}$$
$$+ 5\|A\mathbb{E}_k\|^{14} + \|A\mathbb{E}_k\|^{15}). \quad (2.18)$$

Applying (2.14), which implies that $\|A\mathbb{E}_0\| < 1$, and a similar reasoning as in (2.6)-(2.8), one can obtain.

$$\|A\mathbb{E}_{k+1}\| \leq \frac{1}{32}(\|A\mathbb{E}_k\|^{10} + 5\|A\mathbb{E}_k\|^{11}$$
$$+ 10\|A\mathbb{E}_k\|^{12} + 10\|A\mathbb{E}_k\|^{13}$$
$$+ 5\|A\mathbb{E}_k\|^{14} + \|A\mathbb{E}_k\|^{15})$$
$$\leq \|A\mathbb{E}_k\|^{10} \leq \|A\|^{10}\|\mathbb{E}_k\|^{10}. \quad (2.9)$$

Finally, using the properties of the Moore-Penrose inverse $A^{\dagger}$ and Lemma 2.2, it would be now easy to find error inequality of the new scheme (2.2) as follows:

$$\|V_{k+1} - A^{\dagger}\| = \|A^{\dagger}AV_{k+1} - A^{\dagger}AA^{\dagger}\|$$
$$\leq \|A^{\dagger}\|\|AV_{k+1} - AA^{\dagger}\|$$
$$= \|A^{\dagger}\|\|A\mathbb{E}_{k+1}\|$$
$$\leq \|A^{\dagger}\|\|A\|^{10}\|\mathbb{E}_k\|^{10}. \quad (2.20)$$

Thus $\|V_k - A^{\dagger}\| \to 0;$ that is, the sequence of (2.2) converges to the Moore{Penrose inverse in tenth order as $k \to \infty$. This ends the proof.

III. COMPUTATIONAL EFFICIENCY

Let us consider the following computational efficiency index as given by Traub in Appendix C of [19]:

$$C.E.I = p^{\frac{1}{c}}, \quad (3.1)$$

whereas C stands for the total computational cost of an algorithm and p is the local convergence order.

It is clear that the most impressive cost per cycle of each Schulz-type method is matrix by matrix multiplications. Let us assume that the cost of mmms be unity (as Traub made in [19]). Then the computational efficiency index with η number of mmms per step becomes

$$C.E.I = p^{\frac{1}{\eta s}}, \quad (3.2)$$

where s is the number of iterations (steps) that an iterative algorithm requires to converge.

Soderstrom and Stewart in [12] illustrated that the approximate number of iterations that the Schulz scheme (1.1) requires in a

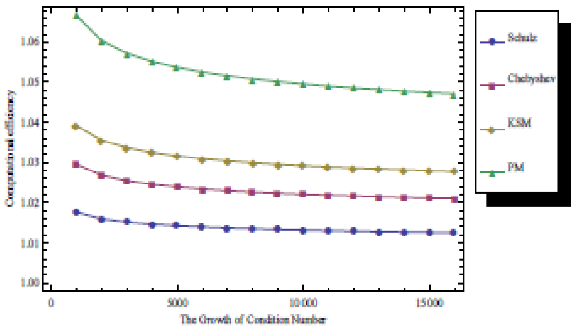

Fig. 1 The comparison of computational efficiency indices for different methods.

machines precision to coverage is given by

$$s \approx 2log_2\kappa_2(A), \qquad (3.3)$$

where k_2 denotes the condition number of the matrix A in norm 2. Hence, similar to (3.3) under the same conditions, the approximate required number of iterations, for a pth-order iterative method to converge [15] is given by

$$s \approx 2log_p\kappa_2(A). \qquad (3.4)$$

Therefore, the computational efficiency index of a pth-order matrix iterative method with η number of matrix by matrix multiplications per cycle would become

$$CEI \approx p^{\frac{1}{\eta(2\log_p \kappa_2(A))}}. \qquad (3.5)$$

Using this index, a comparison has been made in Fig.1 of the iterative algorithms(1.1), (1.2), (1.3) and (2.2) denoted by "SM", "CM", "KSM" and "PM", respectively. Fig.1 reveals that by growth of the condition numbers, the computational efficiency of all methods decreases. But here our proposed algorithm shows its dominancy in terms of the computational efficiency.

Remark 3.1. The new iteration method (2.2) possesses tenth-order of convergence using only seven matrix by matrix multiplications, while the schemes (1.1), (1.2) and (1.3) reach 2nd, 3rd, and 10th orders, respectively, by consuming 2, 3, and 10 matrix by matrix multiplications. Therefore, if one applies the definition of informational efficiency index as $IIEI = \frac{p}{\eta}$, in which ρ stands for the local order of convergence and η is the number of matrix by matrix multiplications per computing step, then the reported method (2.2) achieves the efficiency $\frac{10}{7} = 1.4286$, which beats its other competitors, $\frac{2}{2} = 1$ of (1.1), $\frac{3}{3} = 1$ of (1.2) and $\frac{10}{10} = 1$ of (1.3).

This reveals that the iterative process (2.2) reduces the computational complexity by using less number of basic operations and leads to the better equilibrium between the high speed and the operational cost.

IV. NUMERICAL EXPERIMENTS

We herein present some numerical tests to illustrate the efficiency of the suggested method to compute the Moore-Penrose inverse. For numerical comparisons, we have used the methods "SM", "CM", "KSM" and "PM", in double precision arithmetic. The computer specifications are Microsoft 7 Windows 7 Ultimate Intel(R), Core(TM)2 Duo CPU E7200 @ 2.53 GHz, with 2 GB of RAM.

Example 1. This experiment is devoted to the applicability of our proposed method for finding the Moore-Penrose inverse of the following 20 large random sparse complex matrices of the size $m \times n = 1400 \times 1800$ (wherein $I = \sqrt{-1}$):

```
m = 1400; n = 1800; number =
20; SeedRandom[123];
Table[A[l] =
SparseArray[{Band[{200, 1000},
{m, n}] →
Random[] - I, Band[{20, 70},
{m, n}] → {1.1, -Random[]},
Band[{-300, 900}] → -3.02,
Band[{450, -70}] → 0.1 I},
{m, n}, 0.];, {l, number}];
threshold = 10^-8;
```

in machine precision with stopping criterion $\|V_{k+1} - V_k\|_\infty \leq 10^{-6}$.

In this example, the initial approximations computed in accordance to the Theorem 2.3 for each random test matrix and is written as V0= ConjugateTranspose [A[j]]* (1./((SingularValueList[A[j],1][[1]])^2)) in our MATHEMATICA codes. As the programs were running, we calculated the rnning time using the command Absolute Timing [] to report the elapsed computational

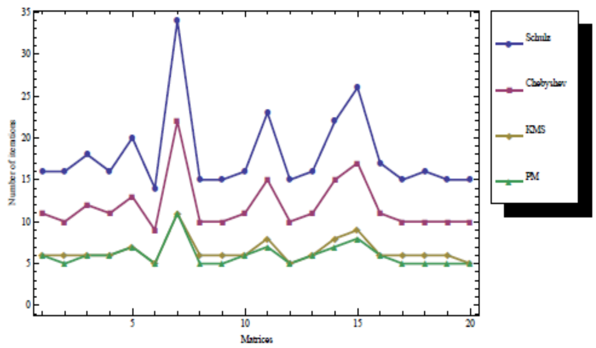

Fig.2 The results of comparisons in terms of number of iterations for Example 1.

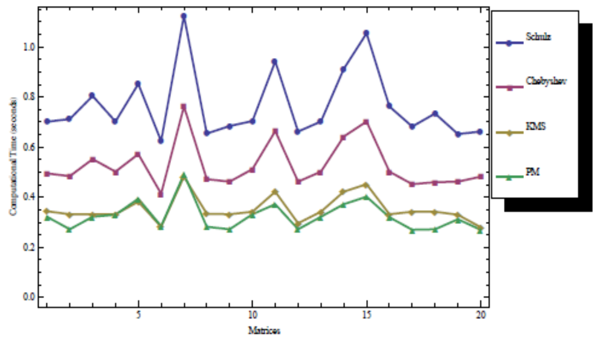

Fig.3 The results of comparisons in terms of computational time for Example 1.

time (in seconds) for the experiment. Here, we also defined the identity matrix by Id =

SparseArray[{{i_,i_}→1.}, {m,m}, 0.]. while maximum number of iterations is set to 100.

The results of comparisons in terms of number of iterations and elapsed computational time are reported in Fig.2 and Fig.3, respectively. The attained results reverify the robustness of the proposed iterative method (2.2) by a clear reduction in the number of iterations and the elapsed time.

V. CONCLUSION

In this work, we have developed a new iteration scheme for finding the matrix inversion and then extended it for Moore-Penrose generalized inverse. It has been proved that the method attains tenth order by consuming only seven matrix by matrix multiplications per iteration. Hence, it possesses higher informational efficiency index than the other existing methods in this literature, which makes it efficient in finding the Moore-Penrose generalized inveres. Latter fact is additionally confirmed by numerical experiments.

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