

A Review of q-deformation in Nonlinear Maps

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Abstract –A review of q-deformed maps is presented in the following text. A general introduction to nonlinear maps is given. Quantum group which leads to the q-deformation process of nonlinear dynamical system is touched upon. A q-deformed Gaussian map q-deformed Logistic map and q-deformed Henon map is discussed with the plots of Gaussian map and its q-deformed version, Logistic map and its q-deformed version.

Index Terms – Hopf Algebra, Nonlinear Map, q-Deformation, Quantum Group.

I. INTRODUCTION

Maps in nonlinear dynamical systems are known variously as difference equations, recursion relations, iterated maps, or simply maps. For instance suppose we repeatedly pressed the cosine button on the calculator, starting from some number x_0 . Then the successive readouts are $x_1 = \cos x_0$, $x_2 = \cos x_1$ and so on.[1]Set your calculator to radian mode and try it. Can you explain the result that emerges after many iterations? The rule $x_{n+1} = \cos x_n$, is an example of a one-dimensional map, so called because the points x_n belong to the one dimensional space of real numbers. The sequence x_0, x_1, x_2, \dots is called the orbit starting from x_0 .

Maps arise in various ways:

1. As tools for analyzing differential equations. We have already encountered maps in this role. For instance, Poincare' maps allowed us to prove the existence of a periodic solution for the driven pendulum and Josephson junction, and to analyze the stability of periodic solutions in general that the Lorenz map provided strong evidence that the Lorenz attractor is truly strange, and is not just a long-period limit cycle.

2. As models of natural phenomenon. In some scientific contexts it is natural to regard time as discrete. This is the case in digital electronics, in parts of economics and finance theory, in impulsively driven mechanical systems, and in the study of certain animal populations where successive generations do not overlap.

3. As simple examples of chaos. Maps are interesting to study in their own right, as mathematical laboratories for chaos. Indeed, maps are capable of much wilder behavior than differential equations because the points x_n hop along their orbits rather than flow continuously.

II. QUANTUM GROUPS AS PRECURSOR TO q-DEFORMATION

There is an emergence in the studies of quantum group recently leading to analysis of several q-deformed physical systems. Theory of quantum groups turned the attention of physicists to the rich mathematics of q-series, q-special functions, etc., with a history going back to the nineteenth century [2]. Here we suggest a scheme of q-deformation using nonlinear maps.

But first let's take look at quantum groups in brief. A quantum group is a kind of non-commutative algebra with additional structures. In general, a quantum group is some kind of Hopf algebra [3]. There is no single, all-encompassing definition, but instead a family of broadly similar objects. Let G be a group in the usual sense. Meaning a set satisfying the group axioms, and k be a field. With this group one can associate a commutative, associative k -algebra of functions from G to k with point wise algebra structure, i.e. for any two elements f and f' , for any scalar $\alpha \in k$ and $g \in G$ we have,

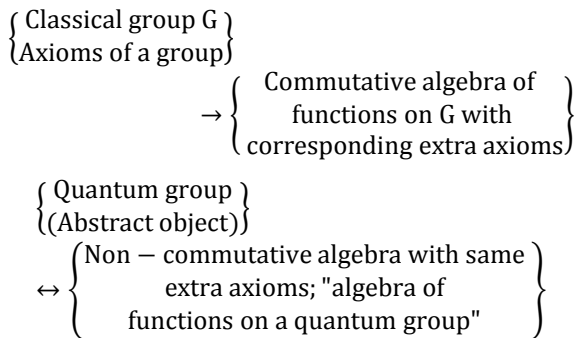
$$(f + f')(g) := f(g) + f'(g)$$

$$(\alpha f)(g) := \alpha f(g)$$

$$(ff')(g) := f(g)f'(g)$$

If G is a topological group, usually only continuous functions are considered and for an algebraic group the functions are normally polynomial functions. These algebras are called "algebras of functions on G ." These algebras inherit some extra structures and axioms for those structures from the group structures and its axioms on G . Locally compact groups can be reconstructed from this algebra. Now the algebra is deformed or quantized, i.e. the algebra structure is changed so that the algebra is called "algebra of functions on a quantum group", where "quantum

group” is just an abstract object ‘described’ by the deformed algebra. This process can be summarized as follows:



There is a similar concept of “quantum spaces”: If G acts on a set X (e.g. a vector space), there is a corresponding so called coaction of the commutative algebra of functions on G on the commutative algebra of functions on X satisfying certain axioms. The latter algebra can often be deformed or quantized into a non commutative algebra, called the “algebra of functions on a quantum space” with a similar coaction. There are three ways of considering algebras of functions on a group and their deformations:

- (a) Polynomial functions $\text{Poly}(G)$ (developed by Woronowicz and Drinfel’d)
- (b) Continuous functions $C(G)$, if G is a topological group (developed by Woronowicz)
- (c) Formal power series (developed by Drinfel’d)

The notion of deformation is not uncommon in physics. Quantum mechanics may be viewed as a deformation of classical mechanics, the deformation parameter being \hbar . Relativistic kinematics can be looked upon as deformation of Newtonian kinematics. Here, $\beta = v/c$ plays the role of deformation parameter. In a similar vein, quantum algebras are nonlinear extension of classical Lie algebras i.e. in general, the commutators between generators are no longer a linear combination of the generators of the set. These algebras contain one or more deformation parameters, such that in the appropriate limit of these parameters, quantum algebras go back to classical Lie algebras. The word ‘quantum’ was initially used to denote these algebras owing to analogy with the above mentioned limiting relation between classical and quantum mechanics. The relation between quantum groups and their algebras is much like the same, as between Lie groups and Lie algebras. Thus quantum algebra is generated by infinitesimal generators of the corresponding quantum group. In case of a classical Lie group, its elements can be obtained by an exponential map from the Lie algebra generators with a set of abelian group parameters. Whereas, in case of quantum groups, we have a similar relationship between quantum group and quantum algebra, only that the group parameters become non-commuting among themselves. This situation is similar to the

commuting variables of classical mechanics becoming operators in quantum mechanics. The concept of a quantum algebra (or quantum group) goes back to late seventies. It was introduced independently, by Kulish and Reshitikhin, Sklyanin, Drinfeld and Jimbo under different names, like quantized universal enveloping algebras or (Hopf algebras) and independently by Woronowicz in terms of compact matrix pseudogroup. In the approach of Manin, quantum algebras are identified with linear transformations on quantum planes with non-commutative coordinates. Wess and Zumino developed the differential calculus on non-commutative planes. In the following, we describe briefly the alternative approaches due to Manin to quantum groups. For the connections between different approaches, the reader is referred to literature.

III. WHAT IS A q-DEFORMATION?

In 1846 Heine [4] defined a number to a basic number as,

$$[n]_q = \frac{1-q^n}{1-q} \tag{1}$$

Such that $[n]_q \rightarrow n$ when $q \rightarrow 1$. In 1904 Jackson defined a q-exponential function given by,

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{2}$$

With,

$$[n]_q! = [n]_q [n-1]_q \dots \dots \dots [2]_q [1]_q \tag{3}$$

$$[0]_q! = 1, \tag{4}$$

As the solution of the q-differential equation,

$$\frac{df(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x} = f(x) \tag{5}$$

It is seen that $e_q^x \rightarrow \exp(x)$ in the limit $q \rightarrow 1$ when the Jackson q-differential operator $d/d_q x$ also becomes the usual differential operator d/dx .

The mathematics of quantum groups necessitated a new deformation of number as,

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{6}$$

Which also has the property that in the limit $q \rightarrow 1$, $[n]_q \rightarrow n$. The associated q-exponential function is given by equation (2) but with $[n]_q$ defined according to (5).

In non extensive statistical mechanics of Tsallis [4], a new q-exponential function has been introduced as,

$$e_q^x = (1 + (1-q)x)^{1/(1-q)} \tag{7}$$

Which satisfies the nonlinear equation given by,

$$\frac{df(x)}{dx} = (f(x))^q \tag{8}$$

And has the required limiting behavior: $e_q^x \rightarrow \exp(x)$ when $q \rightarrow 1$. This e_q^x plays a very important role

in the non-extensive statistical mechanics by replacing $\exp(x)$ in certain domains of application. It should be noted that it is natural to define a generalized exponential function as in (6) if we consider the relation,

$$e^x = \lim_{N \rightarrow \infty} \left(1 + \frac{x}{N}\right)^N \quad (9)$$

And regard $1/N$ as a continuous parameter. The non extensive statistical mechanics has found applications in a wide range of physical problems [6], including the study of nonlinear maps at the edge of chaos.

IV. q-DEFORMATION IN GAUSSIAN MAP

Here we will give the example of Gaussian map which is based on the Gaussian exponential function. It is characterized by two parameters b and c as follows,

$$x_{n+1} = e^{-bx^2} + c \quad (\text{Gaussian map}) \quad (10)$$

The q -deformed version of Gaussian map i.e. q -Gaussian map of the above definition of the q -deformation of one dimensional nonlinear map is given as,

$$x_{n+1} = e^{-b\{[x]_q\}^2} + c \quad (q\text{-Gaussian map}) \quad (11)$$

Or,

$$x_{n+1} = e^{-b\left\{\frac{x}{1-(1-q)(x-1)}\right\}^2} + c \quad (q\text{-Gaussian map}) \quad (12)$$

Which is under the limit $q \rightarrow 1$ becomes the original Gaussian map.

Let's see how this happens with plots for different values of q for q -Gaussian map shown below. The successive plots of q -Gaussian map are plotted for $q = 0$ to $q = 1$.

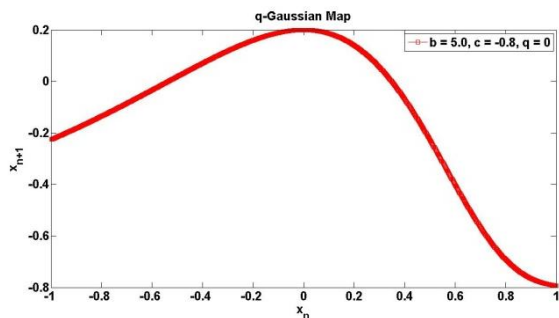


Figure 3.1: q -deformed Gaussian map for $q = 0$

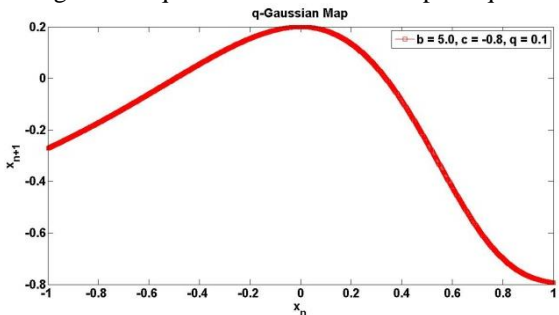


Figure 3.2: q -deformed Gaussian map for $q = 0.1$

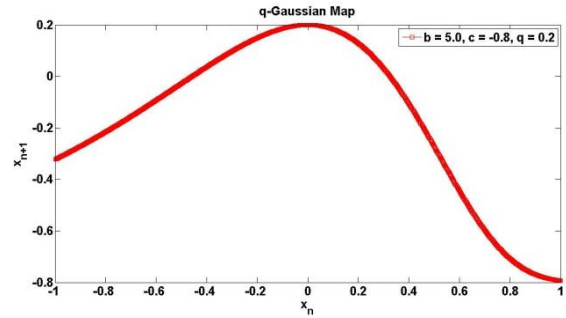


Figure 3.3: q -deformed Gaussian map for $q = 0.2$

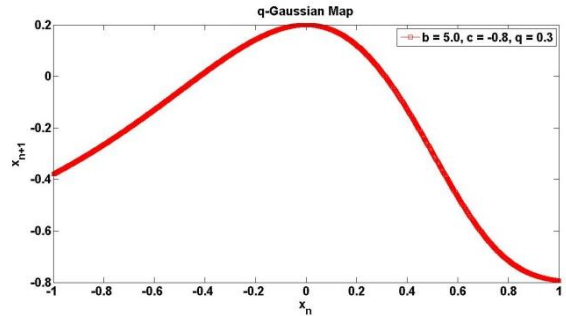


Figure 3.4: q -deformed Gaussian map for $q = 0.3$

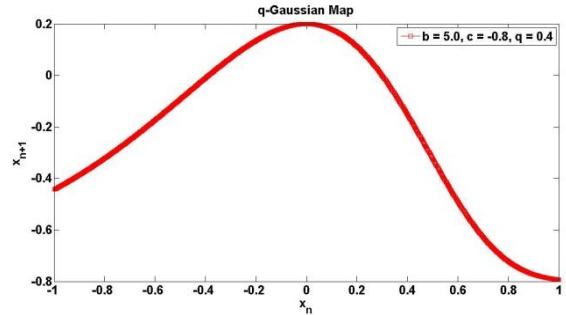


Figure 3.5: q -deformed Gaussian map for $q = 0.4$

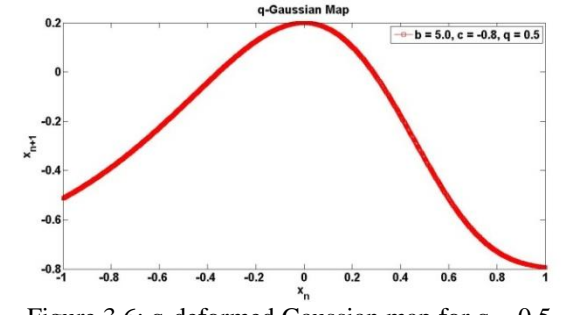


Figure 3.6: q -deformed Gaussian map for $q = 0.5$

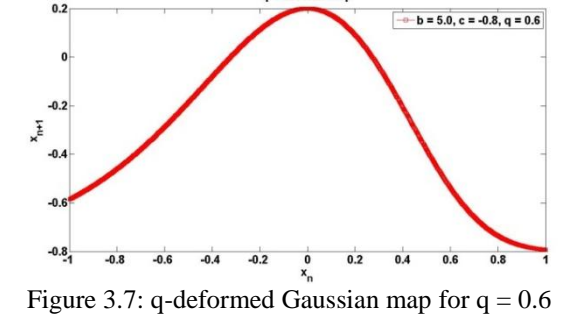


Figure 3.7: q -deformed Gaussian map for $q = 0.6$

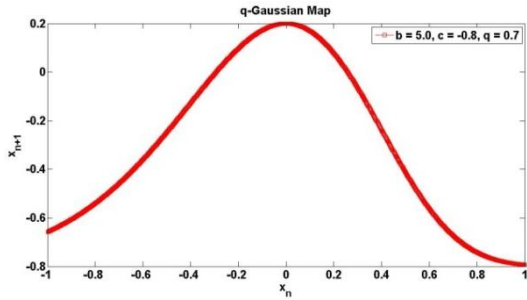


Figure 3.8: q-deformed Gaussian map for q = 0.7

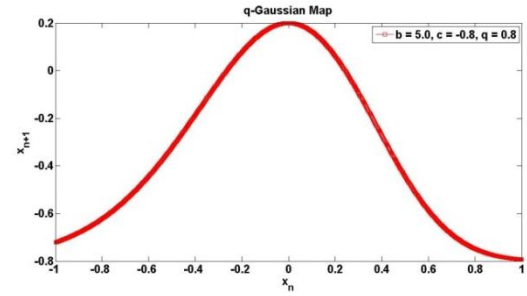


Figure 3.9: q-deformed Gaussian map for q = 0.8

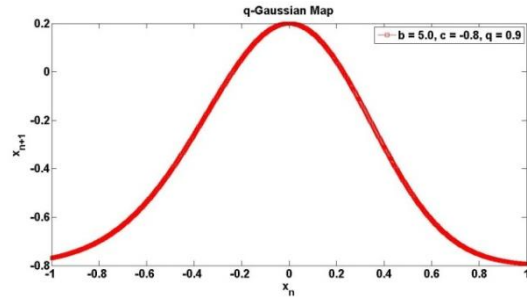


Figure 3.10: q-deformed Gaussian map for q = 0.9

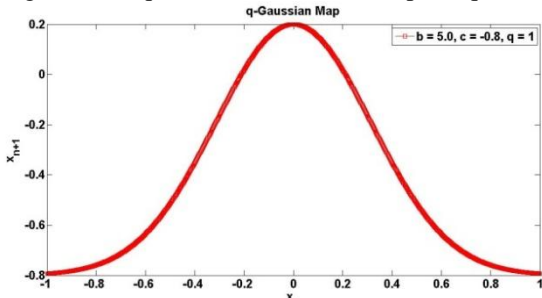


Figure 3.11: q-deformed Gaussian map for q = 1.0

V. q-DEFORMATION IN LOGISTIC MAP

The q-deformation scheme for discrete dynamical maps we suggest is as following. For the one dimensional map,

$$x_{n+1} = f(x_n) \tag{12}$$

The q-deformed version is,

$$x_{n+1} = f([x_n]_q) \tag{13}$$

Where, $[x_n]_q$ is in general, any q-deformed value of x_n ; essentially a q-deformed map is obtained by composing the given map with a basic number deforming map. For example, the q-deformed logistics map defined as follows,

$$F(x) = \frac{a(1+(1-q)x(1-x))}{(1+(1-q(1-x)))^2} \tag{14}$$

For $a = 4$. The usual logistic map corresponds to $q \rightarrow 1$. It is clear that as q moves away from 1, the map gets more skewed. The map is skewed to the right for positive q and to the left for negative q . The lower and upper bounds for q are, respectively -1 and ∞ where $F(x)$ vanishes. Taking the domain of the q-logistic map the same as for the logistic map, namely the interval $[0,1]$, it is found that the range of the q-logistic map is the same as for the logistic map. This is so because over the interval $[0,1]$ the map function $F(x)$ has no singularity for $-1 < (1 - q) < \infty$, and $x \leq [x] \leq 1$ for $-1 \leq (1 - q) < 0$ and $[x] \leq x$ for $(1 - q) \geq 0$. There is one important qualitative difference between the usual logistic map and its deformed version. The deformed map is concave in parts of x-space, whereas the usual logistic map is always convex.

Let's see the plots for different values of q for q-Logistic map shown below. The successive plots of q-Logistic map are plotted for $q = 0$ to $q = 1$.

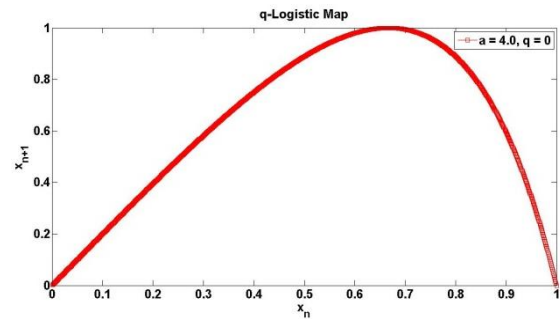


Figure 4.1: q-deformed Logistic map for q = 0

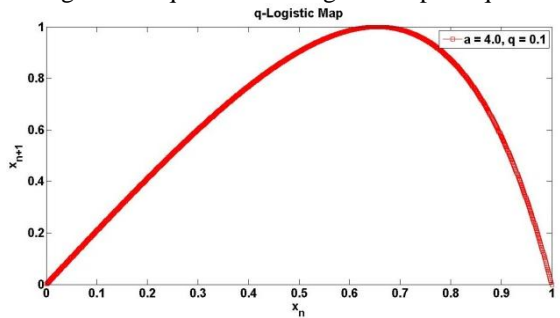


Figure 4.2: q-deformed Logistic map for q = 0.1

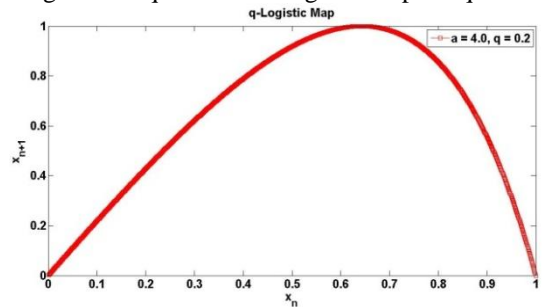


Figure 4.3: q-deformed Logistic map for q = 0.2

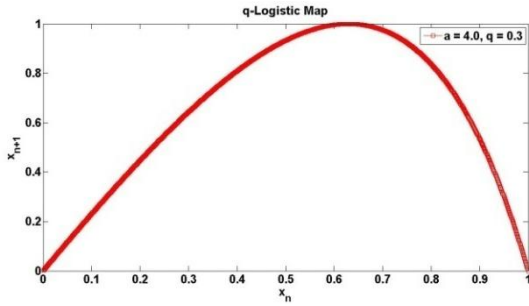


Figure 4.4: q-deformed Logistic map for q = 0.3

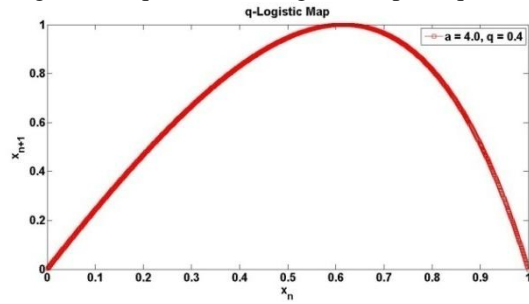


Figure 4.5: q-deformed Logistic map for q = 0.4

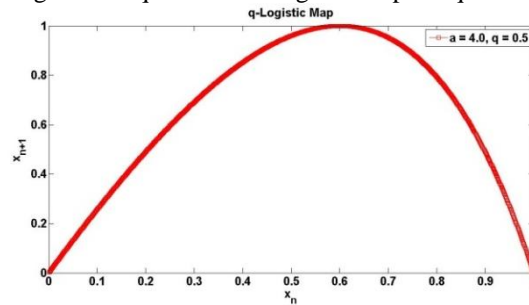


Figure 4.6: q-deformed Logistic map for q = 0.5

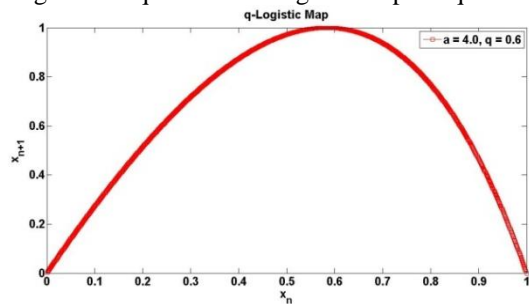


Figure 4.7: q-deformed Logistic map for q = 0.6

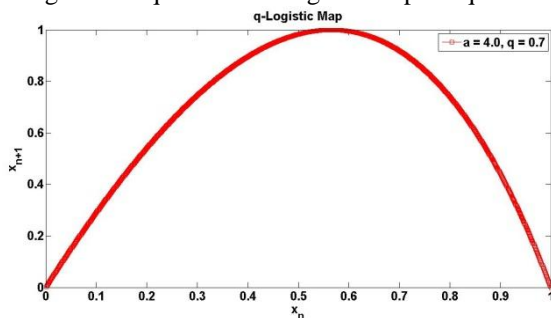


Figure 4.8: q-deformed Logistic map for q = 0.7

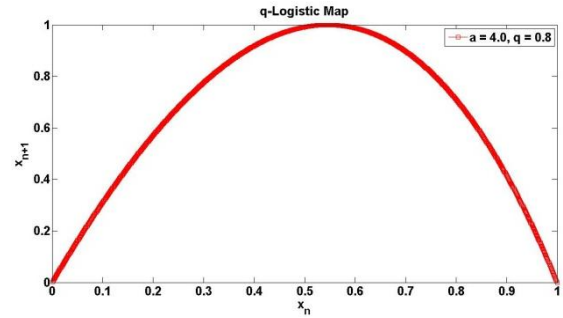


Figure 4.9: q-deformed Logistic map for q = 0.8

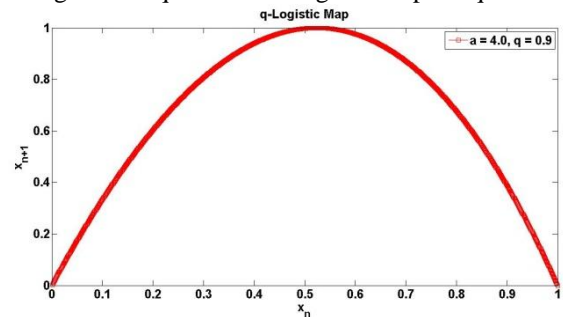


Figure 4.10: q-deformed Logistic map for q = 0.9

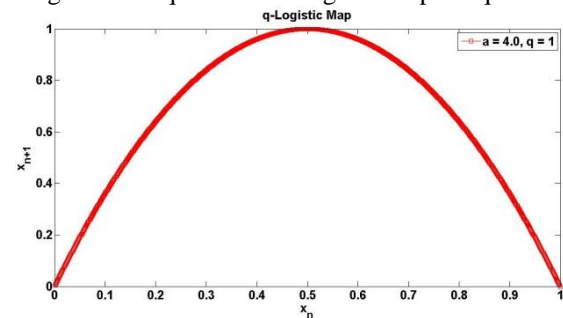


Figure 4.11: q-deformed Logistic map for q = 1.0

VI. q-DEFORMATION IN HENON MAP

In the series of work on q-deformation of various mathematical objects recently, Jaganathan and Sinha [7] have introduced a scheme for the q-deformation of nonlinear maps. They have explained the q-deformation procedure with a simplest smooth 1-D nonlinear map- the Logistic map. They have shown that the q-logistic map exhibits a wide spectrum of interesting behaviors including the coexisting attractors, which are not present in the canonical logistic map. Later Patidar [8]; Patidar&Sud [9]; Patidar et al [10]; have analyzed the dynamical behavior of q-deformed version of another famous 1-D map – the Gaussian map, which is perhaps the only known 1-D map, exhibiting coexisting attractors.

One of the simplest mathematical models, which exhibits strange attractor is the quadratic map introduced by Henon [11]; Lakshmanan & Rajasekar [12]; Hillborn [13]. The mathematical form of the Henon map is given by,

$$x_{n+1} = f(x_n, y_n) = 1 - \alpha x_n^2 + y_n \quad (15)$$

$$y_{n+1} = g(x_n, y_n) = \beta x_n \quad (16)$$

If we generalize the above definition of q-deformation of 1-D nonlinear map to the 2-D case then we may introduce two different state variables q_x and q_y , for two different state variables x_n and y_n , respectively. Such generalized form of a 2-D q-deformed map is given by,

$$x_{n+1} = f\left([x_n]_{q_x}, [y_n]_{q_y}\right) \quad (17)$$

$$y_{n+1} = g\left([x_n]_{q_x}, [y_n]_{q_y}\right) \quad (18)$$

Where,

$$[x_n]_{q_x} = \frac{x_n}{(1-(1-q_x)(x_n-1))} \quad (19)$$

And,

$$[y_n]_{q_y} = \frac{y_n}{(1-(1-q_y)(y_n-1))} \quad (20)$$

Clearly under the limits $q_x \rightarrow 1$ and $q_y \rightarrow 1$ the q-deformed 2-D map reduces to the original map. The explicit form of the q-Henon map (by introducing $1 - q_x = \epsilon_x$ and $1 - q_y = \epsilon_y$), which we use throughout this study is given by,

$$x_{n+1} = 1 - \alpha \left(\frac{x_n}{1 - \epsilon_x(x_n - 1)} \right)^2 + \left(\frac{y_n}{1 - \epsilon_y(y_n - 1)} \right) \quad (21)$$

$$y_{n+1} = \beta \left(\frac{x_n}{1 - \epsilon_x(x_n - 1)} \right) \quad (22)$$

Which is under the limits $\epsilon_x \rightarrow 0$ and $\epsilon_y \rightarrow 0$, becomes the canonical Henon map [equation (17) and (18)].

VII. CONCLUSIONS

We have reviewed the q-deformed nonlinear maps. q-deformation in nonlinear maps was discussed through Gaussian map, Logistic map and Henon map. The q-deformation of Gaussian map and Logistic map is espoused by plotting the maps in MATLAB with changing the q-deformation parameter from 0 to 1. It shows that as the parameter q approaches 1, the original map is recovered. The q-deformation of nonlinear maps find applications in a number of areas, including the study of fractals and multi-fractal measures, and expressions for the entropy of chaotic dynamical systems. The relationship to fractals and dynamical systems results from the fact that many fractal patterns have the symmetries of Fuchsian groups in general and the modular group in particular. The connection passes through hyperbolic geometry and ergodic theory, where the elliptic integrals and modular forms play a prominent role. One more advantage with the study of q-deformed nonlinear maps is that the additionally introduced deformation parameter can be

varied as per our requirements to fit a possible large range of functional forms, which are similar in nature.

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