Developments on Variational Inclusions

Poonam Mishra

Assistant Professor (Mathematics), ASET, AMITY University, Raipur, Chhattisgarh

Abstract - The purpose of this paper is to study the chronicle developments of variational inclusions defined in various spaces. By using the different operator techniques, equivalence between various kinds of variational inclusions and fixed point problems have been established. We have given an account of various features being employed to further extend the field of variational inclusion. We analyze further scope of research in this area of variational inequality.

Mathematics subject classification (2000): 49J40, 90C33.

Keywords and phrases: Variational inequality, Variational inclusion, Resolvent operators, Resolvent Equations.

I. INTRODUCTION

The study of variational inequalities have been extended and generalized in almost every field and many new and innovative techniques have been developed. One of the important and useful generalizations of variational inequality is the field of a generalized mixed type variational inequality which contains a nonlinear term. Due to this non linear term, the projection method cannot be used to study the existence and algorithm of solutions for the generalized mixed type variational inequality. These facts motivated Hassouni and Moudafi [10], in 1994 to suggest the resolvent operator technique which does not depend on the projection. He studied mixed type of variational inequalities, called **variational inclusions**.

Variational inclusions and related problems are being studied extensively by many authors and have important applications in operations research, optimization, mathematical finance, decision sciences and other several branches of pure and applied sciences, see [2] to[37] and the references therein.

Variational inclusions involving three operators are useful and important extensions and generalizations of the general variational inequalities with a wide range of applications in industry, mathematical finance, economics, decision sciences, ecology, mathematical and engineering sciences, see [1] to [37] and the references therein. It is well known that the projection method and its variant forms including the Wiener-Hopf equations cannot be extended and modified for solving the variational inclusions.These facts and comments have motivated to use the technique of the resolvent operators.

This technique can lead to the development of very efficient and robust methods since one can treat each part of the original operator independently. A useful feature of these iterative methods for solving the general variational inclusion is that the resolvent step involves the maximal monotone operator only, while other parts facilitates the problem decomposition. Essentially using the resolvent technique, one can show that the variational inclusions are equivalent to the fixed point problems. This alternative equivalent formulation has played very crucial role in developing some very efficient methods for solving the variational inclusions and related optimization problems, see [15] to [36] and the references therein. Related to the variational inclusions, we have the problem of solving the resolvent equations, which are mainly due to Noor [17], [18], [20].

Essentially using the resolvent operator technique, we can establish the equivalence between the resolvent equations and the variational inclusions. These equivalence formulations are more general and flexible than the resolvent operator method. Resolvent equations technique has been used to suggest and analyze several iterative methods for solving variational inclusions and related problems, see [17] to [23], [32],[35],[36] and the references therein.

Motivated and inspired by the recent research activities in these areas, some new classes of variational inclusions and resolvent equations have been introduced. Essentially using the resolvent operator methods, the equivalence between the resolvent equations and the general variational inclusions is established. This alternative equivalent formulation is used to suggest some iterative methods for solving the general variational inclusions. Since the variational inclusions include the mixed variational inequalities and related optimization problems as special cases, results proved continue to hold for these problems too.

II. DEVELOPMENTS

Hassouni and Moudafi [10], in 1994, suggested the resolvent operator technique for variational inclusion. Then various forms of variational inclusions are introduced and applied by many researchers. Fang and

Huang [6], introduced a new class of monotone operators—H-monotone operators. The resolvent operator associated with an H-monotone operator is defined and its Lipschitz continuity is presented.

Fang and Huang [7], first introduced a new class of generalized accretive operators named H-accretive operators in Banach spaces. By studying the properties of H-accretive operators, the concept of resolvent operators associated with the classical m-accretive operators to the new H-accretive operators is extended. In terms of the new resolvent operator technique, the approximate solution for a class of variational inclusions involving H-accretive operators in Banach spaces is given.

Fang, Cho and Kim [8] and Fang, Huang and Thompson [9] introduced and studied a new system of variational inclusions involving (H, η)-monotone operators in Hilbert space. Using the resolvent operator associated with (H, η)-monotone operators, the existence and uniqueness of solutions for this new system of variational inclusions is proved.

Lan , Cho and Verma [11], introduced a new concept of (A, η)-accretive mappings, which generalizes the existing monotone or accretive operators. Applications of (A, η)-accretive mappings to the approximation-solvability of this class of nonlinear relaxed Cocoercive variational inclusions in q-uniformly smooth Banach spaces is studied.

Zou and Huang [37], introduced a new H(·,·)-accretive operator in Banach spaces and the Lipschitzian continuity of the resolvent operator for the H(·,·)-accretive operator is showed. By using the technique of resolvent operator, an iterative algorithm for solving a class of variational inclusions is constructed.

R. Ahmad and M. Dilshad [1], in 2011, generalized H(.,.) accretive operator introduced by Zou and Huang [37] and is called as H(.,.)- φ - η -accretive operator. The resolvent operator associated with H(.,.)- φ - η - accretive operator is defined and Lipschitz continuity is proved. By using these concepts an iterative algorithm is suggested to solve a generalized variational-like inclusion problem.

Xu and Wang [36], introduced a new class of monotone operators: (H(·, ·), η)- monotone operators, which provide a unifying framework for maximal monotone operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H, η)-monotone operators, A-monotone mappings, (A, η)-monotone operators, G-η-monotone operators, M-monotone operators, H-monotone operators, A-monotone operators and H-η-monotone operators. The resolvent operator associated with an (H(·, ·), η)-monotone operator is defined and its Lipschitz continuity is presented.,

Ahmad et al. [1], introduced a new -cocoercive operator, which generalizes many existing monotone operators , Motivated by works of Ansari and Yao [4], Ahmad and Dilshad [2] using the technique of resolvent operator, constructed an algorithm for solving a variational-like inclusion problem.

In 1998, Noor [21] studied a class of generalized set-valued variational inclusions and resolvent equations. They establish the equivalence between the generalized set-valued variational inclusions, the resolvent equations, and the fixed-point problem, using the resolvent operator technique. This equivalence is used to suggest and analyze some iterative algorithms for solving the generalized set-valued variational inclusions and related optimization problems.

III. IMPORTANT RESULTS

Let K be a nonempty closed and convex set in a real Hilbert space, whose inner product and norm are denoted by $<.,.>$ and $\|.\|$ respectively. Let T, A, g : H → H be three nonlinear operators.

Consider the problem of finding u ∈ H such that $0 \in \rho Tu + u - g(u) + \rho A(u),\ \rho > 0,$ a constant (1)

which is known as the general variational inclusion. Problem (1) is also known as finding the zero of the sum of two (or more) monotone operators..

If A(.) ≡∂ ϕ(.), where ∂ ϕ (.) is the subdifferential of a proper, convex and lower semicontinuous function ϕ: H → R∪ {+∞}, then the problem (1) reduces to finding u ∈ H such that

$$0 \in \rho Tu + u - g(u) + \rho \partial \phi(u),$$

or equivalently, finding u ∈ H such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle + \rho\phi(g(v)) - \rho\phi(u) \geq 0, \quad \forall v \in H \tag{2}$$

The inequality (2) is called the general mixed variational inequality or the general variational inequality of the second kind. It has been shown that a wide class of linear and nonlinear problems arising in various branches of pure and applied sciences can be studied in the unified framework of mixed variational inequalities, see [11] to[13].

We note that if ϕ is the indicator function of a closed convex set K in H, that is,

$$\phi(u) \equiv IK(u) = \phi(u) \equiv IK(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise} \end{cases}$$

then the general mixed variational inequality (2) is equivalent to finding u ∈ K such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K \tag{3}$$

which is called the general variational inequality introduced and studied by Noor [26] in connection with nonconvex functions. See also Noor and Noor [31],[32] for more details.

If $g \equiv I$, the identity operator, then problem (3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u \rangle \geq 0, \ \forall v \in K \quad (4)$$

which is known as the classical variational inequality introduced and studied by Stampacchia [35] in 1964.

We assume that H is a real Hilbert space, C is a nonempty closed and convex subset of H and denote by Fix(T) the set of fixed points of a mapping $T : C \to C$.

Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a multivalued mapping. The so-called quasi-variational inclusion problem (see [22] to [25]) is to find a point $u \in H$ such that

$$\theta \in A(u) + M(u) \quad (5)$$

A number of problems arising in structural analysis, mechanics and economics can be considered in the framework of this kind of variational inclusions.

The set of solutions of the variational inclusion (5) is denoted by Ω.

Special cases Ω

(I) If $M = \partial\varphi : H \to 2^H$, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex and lower semicontinuous function and $\partial\varphi$ is the sub-differential of φ, then variational inclusion problem (4) is equivalent to finding $u \in H$ such that

$$\langle A(u), v-u \rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in H \quad (6)$$

which is called the mixed quasi-variational inequality.

Especially, if $A = 0$, then (6) is equivalent to the minimizing problem of φ on H, i.e., to find $u \in H$ such that $\varphi(u) = \inf_{y \in H} \varphi(y)$.

(II) If $M = \partial\delta_C$, where C is a nonempty closed and convex subset of H and $\delta_C : H \to [0,\infty]$ is the indicator function of C, i.e.,

$$\delta_c(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C \end{cases}$$

then variational inclusion problem (5) is equivalent to finding $u \in C$ such that

$$\langle A(u), v-u \rangle \geq 0, \ \forall v \in C \quad (7)$$

This problem is called Hartman-Stampacchia variational inequality problem.

(III) If $M = 0$ and $A = I - T$ where I is an identity mapping and $T : H \to H$ is a nonlinear mapping, then problem (4) is equivalent to the fixed point problem of T. That is, find $u \in H$ such that $u = Tu$.

Some important notions of variational inclusions are given as:

If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \text{ for all } u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and non expansive, that is, for all $u,v \in H$,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|$$

LEMMA 1.1. For a given $z \in H$, $u \in H$ satisfies the inequality

$$\langle u-z, v-u \rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0,$$

for all $v \in H$, if and only if

$$u = J_\varphi z,$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.

This property of the resolvent operator J_φ plays an important part in developing the numerical methods for solving the mixed variational inequalities.

If the function $\varphi(.)$ is the indicator function of a closed convex set K in H, then it is well known that $J_\varphi = P_K$, the projection operator of H onto the closed convex set K.

Using the definition of the resolvent operator J_A, one can easily prove the following well known result.

LEMMA 1.2. The function $u \in H$ is a solution of the variational inclusion (1) if and only if $u \in H$ satisfies the relation

$$u = J_A[g(u) - \rho Tu],$$

where $\rho > 0$ is a constant and $J_A = (I + \rho A)^{-1}$ is the resolvent operator associated with the maximal monotone operator.

It is clear from Lemma 1.2 that general variational inclusion (1) and the fixed point problems are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems

IV. CONCLUSION:

The research work carried out in the field of variational inclusions has been too vast but still there is considerable scope of further research in this field to develop some new algorithms for solving various variational inclusions. We expect that new researches in this area will not only generalize or extend but also unify the various known results of variational inequalities and still develop some new ideas to find further applications of variational inclusions in various fields.

REFERENCES:

[1] Ahmad, R., Dilshad, M., Wong M.M. and Yao, J.C., 2011. H(.,.)-cocoercive Operator and an Application for Solving Generalized Variational Inclusions, Abs. Appl. Anal. Vol. 2011, Article ID 261534,12 pages.

[2] Ahmad, R. and Dilshad, M., 2012. (H(.,.)-η)-cocoercive Operators and Variational-like Inclusions in Banach Spaces, J. Nonlinear Sci. Appl. 5:334-344.

[3] Ahmad, R., and M. Dilshad, "H(.,.)- φ - η - Accretive Operators and Generalized Variational-Like Inclusions," American Journal of Operations Research, Vol. 1 No. 4, 2011, pp. 305-311.

[4] Ansari, Q.H. and Yao J.C., 2001. Iterative Schemes for Solving Mixed Variational-like Inequalities, J. Optim. Theory Appl.Vol. No-3, 108 : 527-541.

[5] Baiocchi, C.,and Capelo,A.,1984. Variational and Quasi-Variational Inequalities, J. Wiley and Sons, NewYork, London.

[6] Fang, Y.P. and Huang, N.J., 2003.H-Monotone Operator and Resolvent Operator Technique for Variational Inclusions. Appl.Math. Comput. 145 , 795-803.

[7] Fang, Y.P. and Huang N.J.,2004. H-Accretive Operator and Resolvent Operator Technique for Solving Variational Inclusions in Banach Spaces. Appl. Math. Lett. 17 , 647-653.

[8] Fang, Y.P., Cho, Y.J. and Kim J.K.,2004. (H,η)-Accretive Operator and Approximating Solutions for Systems of Variational Inclusions in Banach Spaces.

[9] Fang, Y.P., Huang, N.J. and Thompson, H.B., 2005. A New System of Variational Inclusions with (H,η)-Monotone Operators in Hilbert Spaces. Comput. Math. Appl. 49:365-374.

[10] Hassouni, A. and Moudafi, A.,1994. A Perturbed Algorithm for Variational Inclusions, J. Math. Anal. Appl. 185, 706-712.

[11] Lan,H.Y., Cho,Y.J. and Verma, R.U., 2006.Nonlinear Relaxed Cocoercive Variational Inclusions Involving (A,η)-Accretive Mapping in Banach Spaces, Comput. Math. Appl. 51: 1529-1538.

[12] Moudafi,A.,and Noor,M.,1999. Sensitivity analysis of variational inclusions by the Wiener-Hopf equations technique, J. Appl. Math. Stochastic Anal., **12** , 223–232.

[13] Noor,M.,1999, Some algorithms for general monotone mixed variational inequalities, Math.Computer Modelling, **29**, 7 , 1–9.

[14] Noor, M., 1999.Algorithms for general monotone mixed variational inequalities, J. Math. Anal. Appl., **229** (, 330–343.

[15] Noor, M., 1999.An extraresolvent method for monotone mixed variational inequalities, Math. Computer Modelling, **29** , 95–100.

[16] Noor, M.,1997.Some recent advances in variational inequalities, Part I, basic concepts, New Zealand J. Math., **26** , 53–80.

[17] Noor,M.,1997. Some recent advances in variational inequalities, Part II, other concepts, New Zealand J. Math., **26** , 229-255.

[18] Noor,M.1998. Generalized set-valued variational inclusions and resolvent equations, J. Math. Anal. Appl., **228** (1998), 206–220.

[19] Noor,M, Set-valued mixed quasi variational inequalities and implicit resolvent equations, Math. Computer Modelling, **29** (1999), 1–11.

[20] Noor,M, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., **251** (2000), 217–229.

[21] Noor,M, Equivalence of variational inclusions with resolvent equations, Nonl. Anal., **42** (2000), 963–970.

[22] Noor,M, Three-step iterative algorithms for multivalued quasi variational inclusions, J. Math. Anal. Appl., **255** (2001), 589–604.

[23] Noor,M, A Wiener-Hopf dynamical system for variational inequalities, New Zealand J. Math., **31** (2002), 173–182.

[24] Noor,M, Resolvent dynamical systems for mixed variational inequalities, Korean J. Comput. Appl. Math., **9** (2002), 15–26.

[25] Noor,M Fundamentals of mixed quasi variational inequalities, Inter. J. Pure Appl. Math., **15** (2004), 137–258.

[26] Noor,M, Some developments in general variational inequalities, Appl. Math. Comput., **152** (2004), 199–277.

[27] Noor,M,2008. Differentiable nonconvex functions and general variational inequalities, Appl.Math. Comput., **199** , 623–630.

[28] Noor,M, and Bnouhachem,A.,2007.On an iterative algorithm for general variational inequalities,Appl. Math. Comput., **185** , 155–168.

[29] Noor,M., And Noor,Inayat,K.,2008., On sensitivity analysis of general variational inequalities,Math. Comm., **13** , 75–83.

[30] Noor,M., And Noor,Inayat,K.,2008., Projection algorithms for solving a system of general variational inequalities, Nonl. Anal. .

[31] Noor,M., And Noor,Inayat,K.,2008. Three-step iterative methods for general variational inclusions in Lp -spaces, J. Appl. Math. Computing, **27** (2008), 281–291.

[32] Noor,M., And Noor,Inayat,K.,1997., Multivalued variational inequalities and resolvent equations, Math. Computer Modelling, **26**, 109–121.

[33] Petrot and Balooee: General nonconvex variational inclusions and fixed point problems. Journal of Inequalities and Applications 2013 2013:377.

[34] Shelmas, Eman, Al., 2009.Resolvent Operator Method for general Variational Inclusions. Journal of Mathematical Inequalities.Vol. 3, No-3,455-462.

[35] Stampacchia, G., 1964. Formes Bilineaires Coercitives sur les Ensembles Convexes. C.R. Acad. Sci. Paris 258: 4413–4416

[36] Xu, Z. and Wang, Z.,2010. A Generalized Mixed Variational Inclusion Involving (H(.,.),η)-Monotone Operators in Banach Spaces. J. of Math. Research, Vol.2, No-3:47-56.

[37] Zou,Y.Z. and Huang, N.J., 2008.H(.,.)-Accretive Operator with an Application for Solving Variational Inclusions in Banach Spaces. Appl. Math. Comput. 204:809-816.