Contraction Mapping and Their Generalizations

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***Abstract:-** The contraction mapping principle is one of the most useful tools in the study of nonlinear equations, be they algebraic equations, integral or differential equations. The principle is a fixed point theorem which guarantees that a contraction mapping of a complete metric space to itself has a unique fixed point which may be obtained as the limit of an interaction scheme defined by repeated images under the mapping of an arbitrary starting point in the space. As such, it is a constructive fixed point theorem and hence may be implemented for the numerical computation of the fixed point.*

***Keywords:** Contraction mapping, fixed point theory, compatible mapping.*

I. INTRODUCTION AND THEORY:

Let us first define some terms

Fixed point

Let (x, d) be an metric space & let T: X $\longrightarrow$ X be a mapping. Than T is said to have a fixed point if there exists a point in X such that

$$T(X) = x$$

Contraction mapping

The mapping T is said to be a contraction mapping if

d (Tx, Ty) $\leq$ C d (x,y) $\forall$ x,y ε X

Theorem : Banach's Fixed Point Theorem

Every contraction mapping on a complete metric space has a unique fixed Point.

Proof: Pick an element X_0 belonging to X arbitrarily. Construct the sequence

$\{X_n\}$ as follows:

X_1 = TX_0, X_2 = TX_1 ,

X_3 = TX_2, X_4 =TX_3

----------X_n =TX_{n-1}

This sequence $\{X_n\}$ is called the iterated sequence

We have,

d (X_1, X_2) = d (TX_0, TX_1) $\leq$ cd (XO, X_1) = c`d (XO, TX_o)

d (X_2 , X_3) = d (TX_1, TX_2) $\leq$ cd (X1,X_2) = C^2 d (XO, TX_o)

d (X_3,X_4) = d (TX_2,TX_3) $\leq$ cd (X_2,X_3) = C^3d (X_o,TX_o)

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d (X_n, X_{n+1}) $\leq$ C^n d ,(xo, TX_o)

For any positive integer m

d (X_n,X_m) $\leq$ d (X_n,X_{n+1})+ d (X_{n+1},X_{n+2})+ d (X_{n+1}, X_{n+2}) +

--------------------------------+ d (X_{m-1}, X_m)

$\leq$ C^n d (X_o, TX_o)+ C^{n+1} d (X_o,TX_o)+

------------------+C^{m-1} d (X_o, TX_o)

$\leq$ (C^n+ C^{n+1} +----------- +C^{m-1}) d (X_o, TX_o)

$$= \frac{C^n - C^m \, d(X_o, TX_o)}{1-c}$$

$$< \frac{C^n}{1-c} d(X_o, TX_o)$$

Since $O \leq c \leq 1$

Therefore d $(X_n, X_m) \to 0$ as $n \to \infty$

Thus (X_n) is a cauchy sequence Since, x is complete, (X_n) converges to x.Let us prove that x is a fixed point of the operator T,

i.e. T(x) = x

Now,

d (x, Tx) $\leq$ d(x , X_n) + d(X_n , TX)

$\leq$ d(X, X_n) + d(TX_{n-1}, Tx)

$\leq$ d (x, x_n) + c d (X_{n-1} , X)

$\to$ 0 as n $\to$ ∞ (since lim $\to \infty$ x_n = x)

Hence T (x) = x i.e., x is a fixed point of T.

Uniqueness of T

Let y be an element of X such that .

T (y) = y

Now,

d(x, y) = d (Tx, Ty) $\leq$ cd (x, y)

If, $x \neq y$. than d (x,y) $>$ o

& dividing by d (x, y)

We have C$\geq$ 1, which contradicts the hypothesis that $0 \leq c \leq 1$.

This contraction shows that x = y. Thus the fixed point is unique.

II. RESULTS AND DISCUSSION:

Every compatible pair of mapping is a compatible pair of mapping of type (T).

Proof: Suppose that I & T are compatible mappings of a normed space X into itself. Let $\{x_n\}$ be a sequence in X such that $lim_{n\to\infty}$ Ix_n = $lim_{n\to\infty}$ Tx_n = t some t $\in$ X

We have

$\|$ Ix_n- Ix_n $\| \leq \|$ ITx_n- TIx_n $\|$ + $\|$ TIx_n-Tx_n $\|$ + $\|$ Tx_n - Ix_n $\|$

i.e :

$\|$ ITx_n - Ix_n $\|$ + $\|$ ITx_n - TIx_n $\|$

< 2 $\|$ JTx_n - TJx_n $\|$ + $\|$ TIx_n - Tx_n $\|$ + $\|$ Tx_n - Ix_n $\|$

Letting n→ ∞, since l & T are compatible we have.

$lim_{n\to\infty}$ $\|$ ITx_n - Ix_n $\|$ + $lim_{n\to\infty}$ $\|$ ITx_n - TIx_n $\|$

< $lim_{n\to\infty}$ $\|$ TIx_n - Tx_n $\|$

Therefore, I & T are compatible mapping of type (T).

Similarly, we can show that I & T are compatible mappings of type (l).

Suppose that I & T are compatible mappings of a normed Space X into itself. They are compatible if and only if They are compatible both type (T) and (I).

Proof:

The necessary condition follows by proposition (1.1).

To prove the sufficient condition.

let I & T be compatible of both types (I) & (T). then we have

(1) $lim_{n\to\infty}$ $\|$ ITx_n - Ix_n $\|$ + $lim_{n\to\infty}$ $\|$ ITx_n - TIx_n $\|$ < $lim_{n\to\infty}$ $\|$ TIx_n -Tx_n $\|$

(2) $lim_{n\to\infty}$ $\|$ TIx_n - Tx_n $\|$ + $lim_{n\to\infty}$ $\|$ ITx_n-TIx_n $\|$ < $lim_{n\to\infty}$ $\|$ ITx_n- Ix_n $\|$,

Whenever there exists a sequence $\{x_n\}$ in X such that

$lim_{n\to\infty}$ Ix_n = $lim_{n\to\infty}$ Tx_n = t

for some t in X. Adding (1) & (2) and cancelling the common

term, we obtain

$lim_{n\to\infty}$ $\|$ITx_n - TIx_n] $\| \leq 0$

Which implies that $lim_{n\to\infty}$ $\|$ITx_n - TIx_n $\|$ = 0

Therefore, the mappings I & T are compatible

III. ACKNOWLEDGEMENT:

The author is thankful to Dr. G. K. Goswami (H.O.D. Mathematics & Research centre, Kalyan College, Bhilai) for preparing this paper.

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